W14/2 Group Completions

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Group Completion via Cauchy Sequences:

Let G be an ab. top. group. Assume G is first-countable ($\Leftrightarrow$ 0 has a countable neighborhood basis; this ensures: if $A \subseteq G$, then $x \in \bar{A} \Leftrightarrow \exists$ sequence $(x_n)_{n \geq 0}$ in A s.t. $(x_n)_{n\geq 0} \to x$).

- A sequence $(x_n)_{n\geq 0}$ in G is a Cauchy sequence if
  $\forall U$ open nbhd of 0 $\exists N \geq 0$ $\forall m,n \geq N$: $x_m - x_n \in U$.
- Equivalence: $(x_n)_{n\geq 0} \sim (y_n)_{n\geq 0}$ for Cauchy sequences $\Leftrightarrow$ $(x_n - y_n)_{n \geq 0} \to 0$
- Let $\hat{G}$ be the set of equivalence classes of C.S. modulo $\sim$.
  Elementwise addition & inverses induce a topological group structure on $\hat{G}$. [Topology: $\hat{G}$ is a quotient of the product space $G^{\mathbb{N}_0}$]

Def: If G is a first-countable top. group, then $\hat{G}$ is its completion

Basic Properties: (w/o proofs) Let G, H, K be first-countable, ab. top. groups.

1) $\varphi: G \to \hat{G}$, $a \mapsto (a,a,a,\dots)$ is a continuous group hom.
2) If $f: G \to H$ is a cont. hom., it induces a cont. hom $\hat{f}: \hat{G} \to \hat{H}$.
3) If $f: G \to H$, $g: H \to K$ are cont. homs, then $\widehat{g \circ f} = \hat{g} \circ \hat{f}$ and $\widehat{id_G} = id_{\hat{G}}$, so completion is functorial.
4) $\operatorname{Ker}(\varphi: G \to \hat{G}) = \bigcap \{N : N \text{ nbhd of } 0\} \overset{L9.3}{=} \overline{\{0\}}$

Def: G is complete if the natural map $\varphi: G \to \hat{G}$ is an isomorphism.

($\Rightarrow G$ Hausdorff by L9.3)

Linear Topologies:

Def: Let G be an abelian top. group [A-module]. The topology on G is linear if 0 has a countable nbhd basis consisting of is linear if 0 has a countable nbhd basis consisting of subgroups [submodules]

i.e. $\exists$ a filtration $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots$ by subgroups [submodules] s.t. every nbhd N of 0 contains some G_i.

Properties: (1) Each G_i is open and closed.

[G_i open: $\forall g \in G_i$: $g + G_i$ is a nbhd of g, $g + G_i \subseteq G_i$

G_i closed: $G \setminus G_i = \underbrace{\bigcup_{g \in G \setminus G_i} g + G_i}_{\text{open}} \Rightarrow G_i$ closed.]

(2) If $G_0 \supseteq G_1 \supseteq \dots$ is any filtration by subgroups [submodules], then it induces a topology on G where the G_i form a basis of nbhds of 0 [$\{g + G_i : g \in G, i \geq 0\}$ is a basis for the open sets:

if $j \geq i$: $(g + G_i) \cap (h + G_j) = \left(\bigcup_{h' \in G_i} g + h' + G_j \right) \cap (h + G_j)$

$= \begin{cases} h + G_j, \text{ or} \\ \emptyset \end{cases}$]

W.r.t this topology, G is an abelian top. group. [Exc.]

(3) The quotient topology on each G/G_i is discrete by (1).

Prop 9.4: If G is an ab. group w. linear. top induced by $G_0 \supseteq G_1 \supseteq \dots$, then $\hat{G} \cong \varprojlim_i G/G_i$

Proof: (Sketch) Let $(x_n)_{n \geq 0}$ be a CS. For $i \geq 0$, $(x_n + G_i)_{n \geq 0}$ in G/G_i is eventually constant, say equal to $\xi_i + G_i$. Then $\xi_{i+1} + G_i = \xi_i + G_i$, so $(\xi_i + G_i)_{i \geq 0}$ is a compatible seq., hence $(\xi_i + G_i)_{i \geq 0} \in \varprojlim_i G/G_i$.

If $(x'_n)_{n \geq 0} \sim (x_n)_{n \geq 0}$, then $(x'_n)_{n \geq 0}$ gives rise to the same sequence

If $(x_n')_{n\geq 0} \sim (x_n)_{n\geq 0}$, then $(x_n')_{n\geq 0}$ gives rise to the same sequence because $\forall i \geq 0$: $x_n' - x_n \in G_i$ for large n.

We get a group hom. $\varphi : \hat{G} \to \varprojlim_i G/G_i$.

<u>Inverse Map</u>: Let $(\xi_i + G_i)_{i\geq 0} \in \varprojlim_i G/G_i$, so $\xi_j - \xi_i \in G_i$ $\forall j \geq i \geq 0$.

For $i \geq 0$, let $x_i := \xi_i$. Then $(x_i)_{i\geq 0}$ is a C.S., and taking its equivalence class yields the inverse map to φ [Small details omitted]

<u>Check</u>: These maps are continuous [Topology on $\varprojlim_i G/G_i$ is the subspace top. of the product top. on $\prod_{i\in\mathbb{N}_0} G/G_i$ with G/G_i discrete. Equivalently, it is the initial topology w.r.t. the system maps $\varprojlim_i G/G_i \to G/G_i$.]

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